

# STATISTICAL PROPERTIES OF GENERALIZED VIANA MAPS

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**ABSTRACT.** We study quadratic skew-products with parameters driven over piecewise expanding and Markov interval maps with countable many inverse branches, a generalization of the class of maps introduced by Viana [Vi97]. In particular we construct a class of multidimensional non-uniformly expanding attractors that exhibit both critical points and discontinuities and prove existence and uniqueness of an SRB measure with stretched-exponential decay of correlations, stretched-exponential large deviations and satisfying some limit laws. Moreover, generically such maps admit the coexistence of a dense subset of points with negative central Lyapunov exponent together with a full Lebesgue measure subset of points which have positive Lyapunov exponents in all directions. Finally, we discuss the existence of some non-uniformly hyperbolic attractors for skew-products associated to hyperbolic parameters.

## 1. INTRODUCTION

Since the 1960's, when the concept of uniform hyperbolicity was coined by Smale in [Sm67], a relevant question in dynamical systems is to construct examples that exhibit the hyperbolic features described by the theory. In fact, Hunt and Mackay [HM03] proved that uniformly hyperbolic dynamical systems, among which Smale's horseshoe is a paradigmatic example, arise naturally in physical systems. On other direction, simple examples arising from populational dynamics led to consider the quadratic family  $T_a(x) = ax(1-x)$  or equivalently  $f_a(x) = 1 - ax^2$  that despite the simple formulation presents very rich and complex dynamics. On the one hand it follows from pioneering works by Jakobson, Benedicks and Carleson [Jak81, BC85] that there exists a positive Lebesgue measure set  $\Delta \subset (0, 2]$  such that  $f_a(x) = 1 - ax^2$  has an absolutely continuous ergodic probability measure with positive Lyapunov exponent for all  $a \in \Delta$ . On the other hand, it follows from Graczyk and Świątek [GS97] and Lyubich [Lyu97] that  $f_a$  is hyperbolic for an open and dense set of parameters  $a \in (0, 2]$ . This illustrates that strict non-uniform hyperbolicity is not robust among the family of quadratic maps. In fact, this general one-dimensional feature was established by Kozlovski, Shen and van Strien [KSvS07a] that proved that hyperbolicity is open and dense among  $C^k$  maps of the interval or the circle. We refer the reader to Subsection 3.1 for details.

The class of maps of the cylinder known as *Viana maps* was introduced in [Vi97]: they are any small  $C^3$ -perturbations  $\varphi$  of the skew-product transformations

$$\begin{aligned} \varphi_\alpha : S^1 \times I &\rightarrow S^1 \times I \\ (\theta, x) &\mapsto (g(\theta), f_\alpha(\theta, x)) \end{aligned}$$

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where  $g$  is an expanding map of the unit circle  $S^1$  with  $|g'(\theta)| \geq d$  where  $d \geq 16$  and  $f_\alpha(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$  for some small  $\alpha > 0$  and parameter  $a_0 \in (0, 2]$  such that the quadratic map  $h(x) = a_0 - x^2$  is of Misiurewicz type. Despite the presence of a critical region, which is a circle, Viana proved that this class of transformations of the cylinder  $S^1 \times \mathbb{R}$  have positive Lyapunov exponents in every direction, that is

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi^n(\theta, x)v\| > 0$$

for Lebesgue almost every  $(\theta, x)$  and all  $v \in T_{(\theta, x)}(S^1 \times \mathbb{R})$ . These first multidimensional examples with robust non-uniformly expanding behavior exhibit a weak form of hyperbolicity, namely partial hyperbolicity. Building over this, Alves [Al00] proved that there is a unique  $\varphi$ -invariant probability measure absolutely continuous with respect to Lebesgue. Several other recent contributions and extensions include the ones by Gouzel [Gou07] on the skew products with curve of neutral fixed points, by Buzzi, Sester and Tsujii [BST03] for  $C^\infty$ -perturbations of the skew product  $\varphi_\alpha$  with a weaker condition  $d \geq 2$  and later on by Schnellmann [Sc08] that considered  $\beta$ -transformations and by Schnellmann, Gao, Shen [Sc09, GS12] considering Misiurewicz-Thurston quadratic maps as the base dynamics. In some sense these examples present some weak domination condition.

An important challenge in dynamics is to construct multidimensional attractors without dominated splittings but persistence of nonuniformly expanding behaviour in parameter space. Such phenomena might occur in a parametrized family  $F(x, y) = (a(x, y) - x^2, b(x, y) - y^2)$  as proposed by Bonatti, Díaz and Viana in [BDV05]. A simpler but still intricate class of examples correspond to skew-product of quadratic maps where the base dynamics is a nonuniformly expanding quadratic map. A first important contribution was given by Schnellmann [Sc09], that considered a Misiurewicz-Thurston quadratic map as base dynamics. However, since parameters corresponding to Misiurewicz-Thurston quadratic maps have zero Lebesgue measure in the parameter space then the previous question remains open. On the other hand, since nonuniform expansion is well known to be related with inducing schemes and piecewise expanding maps with infinitely many branches, one important motivation to consider skew products of quadratic maps over expanding dynamics with infinitely many branches is to understand if the technique of inducing can be an useful approach to the previous question.

In broad terms our contribution to the theory of nonuniformly hyperbolic dynamics in this paper is to study a class of quadratic skew-products over a Markov expanding map of the interval with at most countably many inverse branches: skew-products  $\varphi_\alpha(\theta, x) = (g(\theta), f_\alpha(\theta, x))$  with  $g$  piecewise expanding Markov map of the unit interval and  $f_\alpha(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$  for some small  $\alpha > 0$  and parameter  $a_0 \in (0, 2]$ . This class of transformations behave much differently maps depending on the parameter  $a_0$ . In one direction, if  $h(x) = a_0 - x^2$  is a Misiurewicz quadratic map then we overcome the difficulty caused by the presence of critical points and infinitely many invertibility branches to prove the existence of positive Lyapunov exponents at Lebesgue almost every point. In particular, there exists a unique absolutely continuous invariant measure with good statistical properties, thus extending the results of [Vi97, Al00]. Moreover, we also prove that generically such transformations admit the coexistence of the full Lebesgue measure set of points which have only positive Lyapunov exponents together with a dense set of points with one positive and one negative Lyapunov exponents, a fact that was unknown

even in the context of Viana maps in [Vi97]. On the complementary direction, if the parameter  $a_0$  is such that the quadratic map  $h(x) = a_0 - x^2$  is hyperbolic then there exists a unique SRB measure and it is hyperbolic with one positive and one negative Lyapunov exponent. Moreover, the complement of the basin of attraction of the SRB measure consists of a Cantor set of lines and the support of the measure exhibits transversality phenomena as in the class of maps studied by Tsujii [Tsu01] and Volk [Vo11]. We refer the reader to the next section for the precise statements.

This paper is organized as follows. In Section 2 we recall some definitions and state our main results. Some preliminary results are given along Section 3, while the proofs of the results are postponed to Sections 4, 5 and 6.

## 2. STATEMENT OF MAIN RESULTS

In this section we present the necessary definitions to state our main results.

**2.1. Setting.** Let  $\mathcal{P} = \{\omega_i\}_{i \in S, S \subset \mathbb{N}_0}$  be an at most countable partition of the unit interval  $(0, 1]$  by subintervals and  $g : (0, 1] \rightarrow (0, 1]$  be a  $C^3$  piecewise differentiable map. We will say that  $g$  is a *Markov expanding* map if  $g(\omega_i) = (0, 1]$  and the restriction  $g|_{\omega_i}$  is a  $C^3$  diffeomorphism with a  $C^3$  extension to the closure for any  $i \in S$ ,  $|g'(\theta)| \geq d \geq 16$  for all  $\theta \in (0, 1]$ , and there exists  $K > 0$  so that  $|g''| \leq K|g'|^2$ . The later is the so called R nyi condition, which is a sufficient condition to obtain the bounded distortion property in Subsection 3.1. Throughout we assume that  $\log |g'| \in L^1(\text{Leb})$ . This is a natural assumption to obtain finite positive Lyapunov exponent for  $g$  and related with the size of smaller intervals of  $\mathcal{P}$ . Now we introduce the family of skew-products of the space  $(0, 1] \times \mathbb{R}$  with countably many inverse branches.

*Definition 2.1.* We say that a piecewise  $C^3$  map  $\varphi : (0, 1] \times \mathbb{R} \rightarrow (0, 1] \times \mathbb{R}$  is a *generalized Viana map* if it is a skew-product given by

$$\begin{aligned} \varphi_\alpha : (0, 1] \times \mathbb{R} &\rightarrow (0, 1] \times \mathbb{R} \\ (\theta, x) &\mapsto (g(\theta), f_\alpha(\theta, x)) \end{aligned}$$

where  $g$  is a piecewise linear Markov expanding map on  $(0, 1]$  and  $f_\alpha(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$  for some  $\alpha > 0$  and parameter  $a_0 \in (0, 2]$ .

The assumptions on the parameter  $a_0$  will be crucial. Recall that the quadratic map  $h(x) = a_0 - x^2$  is Misiurewicz provided that the critical point is pre-periodic repelling. It is not hard to check that there exists an interval  $I_0 \subset [h^2(0), h(0)]$  such that  $\varphi_\alpha((0, 1] \times I_0) \subset (0, 1] \times I_0$  for every  $\alpha > 0$  small enough. Then we define the attractor  $\Lambda = \Lambda(\varphi_\alpha)$  for  $\varphi_\alpha$  by

$$\Lambda(\varphi_\alpha) = \bigcap_{n \geq 0} \varphi_\alpha^n((0, 1] \times I_0)$$

and consider the restriction  $\varphi_\alpha|_\Lambda$ . Let us mention that although it seems reasonable that some other classes of infinitely branched interval expanding maps can be considered as base dynamics without the R nyi assumption some condition on the decay of the size of the partition elements should be necessary (e.g. otherwise could exist SRB measures with finite positive Lyapunov exponent).

Finally, to study perturbations of this class of skew-products we introduce an appropriate topology. Assume, without loss of generality, that  $J = \mathbb{N}_0$ , that  $(\theta_i)_i$  is a strictly decreasing sequence in  $(0, 1]$  and  $\omega_i = (\theta_{i+1}, \theta_i]$  for all  $i \in \mathbb{N}_0$ . Given

$\varepsilon > 0$  we say that  $\varphi : (0, 1] \times \mathbb{R} \rightarrow (0, 1] \times \mathbb{R}$  is  $\varepsilon$ - $C^3$ -close to the skew-product  $\varphi_\alpha$  above if  $\varphi(\theta, x) = (\tilde{g}(\theta), \tilde{f}(\theta, x))$  is a piecewise  $C^3$  map and satisfies:

- (i)  $\varphi((0, 1] \times I_0) \subset (0, 1] \times I_0$  and  $\tilde{g}$  is a Markov expanding map on  $(0, 1]$ ;
- (ii) If  $(0, 1] = \bigcup_{i \in S} (\tilde{\theta}_{i+1}, \tilde{\theta}_i]$  is a Markov partition for  $\tilde{g}$  then the renormalized maps  $R_i g : \omega_i \times \mathbb{R} \rightarrow (0, 1]$  and  $R_i f : \omega_i \times \mathbb{R} \rightarrow \mathbb{R}$  given respectively by

$$R_i g(\theta) = \tilde{g} \left( \tilde{\theta}_{i+1} + \frac{\tilde{\theta}_i - \tilde{\theta}_{i+1}}{\theta_i - \theta_{i+1}} (\theta - \theta_{i+1}) \right)$$

and

$$R_i f(\theta, x) = \tilde{f} \left( \tilde{\theta}_{i+1} + \frac{\tilde{\theta}_i - \tilde{\theta}_{i+1}}{\theta_i - \theta_{i+1}} (\theta - \theta_{i+1}), x \right)$$

satisfy  $\sup_x \|g(\cdot) - R_i g(\cdot, x)\|_{C^3} < \varepsilon$  and  $\|f|_{\omega_i \times \mathbb{R}} - R_i f\|_{C^3} < \varepsilon$ .

Let us make some comments on our assumptions. We will assume for notational simplicity that the partition  $\mathcal{P}$  is preserved under perturbations, in which case  $C^3$  perturbation coincides with the usual notion for interval maps. Condition (i) implies that the perturbed map  $\varphi : (0, 1] \times \mathbb{R} \rightarrow (0, 1] \times \mathbb{R}$  is a skew-product with countably many domains of invertibility over a Markov expanding map. These are natural assumptions if one assumes the base dynamics to be induced map from some one-dimensional nonuniformly expanding map. Condition (ii) implies the domains of invertibility of  $\varphi$  to be close to those of  $\varphi_\alpha$  and that the dynamics in each domain is  $C^3$ -close to the original one. So, these assumptions require the map  $\varphi$  to be close to  $\varphi_\alpha$  from both the topological and the differentiable viewpoints. Notice that in the case that  $\#S = d < \infty$  with all intervals of the same size, the map  $g$  induces a map in  $S^1 = I/\sim$  and we recover the setting of [Vi97].

**2.2. Statement of results.** We are now in a position to state our main results.

**Theorem A.** *Consider the skew-product  $\varphi_\alpha : (0, 1] \times I \rightarrow (0, 1] \times I$  given by  $\varphi_\alpha(\theta, x) = (g(\theta), f_\alpha(\theta, x))$  and such that  $h(x) = a_0 - x^2$  is Misiurewicz. Then there exists  $c > 0$  such that for any small  $\alpha$  it holds*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi_\alpha^n(\theta, x)v\| \geq c > 0$$

for Lebesgue almost every  $(\theta, x)$  and every  $v \in \mathbb{R}^2 \setminus \{0\}$ . Moreover, there exists  $\varepsilon > 0$  such that the same property holds for every  $\varphi$  that is  $\varepsilon$ - $C^3$ -close to  $\varphi_\alpha$ .

As a byproduct of the proof we obtain some estimates on the decay of the first time at which some hyperbolicity is obtained. These are known as hyperbolic times (see [Al00] for some details). In consequence, one can use the works of Araújo, Solano [ArS11] or Pinheiro [Pi11] to build an inducing scheme and deduce the existence of an SRB measure with good statistical properties. Recall that a  $\varphi$ -invariant and ergodic probability measure  $\mu$  is an *SRB measure* if its basin of attraction

$$B(\mu) = \left\{ (\theta, x) \in (0, 1] \times I_0 : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\varphi^j(\theta, x)} \xrightarrow{w^*} \mu \right\}$$

has positive Lebesgue measure. Here we establish not only uniqueness of the SRB measure as we obtain several important statistical properties. Let  $\mathcal{H}_\beta$  be denote the space of  $\beta$ -Hölder continuous observables. We obtain the following:

**Theorem B.** Let  $\varphi_\alpha : (0, 1] \times I \rightarrow (0, 1] \times I$  be a generalized Viana map  $\varphi_\alpha(\theta, x) = (g(\theta), f_\alpha(\theta, x))$  such that  $h(x) = a_0 - x^2$  is Misiurewicz. Then, for any small  $\alpha > 0$ :

- (1)  $\varphi_\alpha$  is topologically mixing;
- (2) There exists a unique  $\varphi_\alpha$ -invariant measure  $\mu_\alpha$  that is absolutely continuous with respect to Lebesgue on the attractor  $\Lambda(\varphi_\alpha)$ ;
- (3)  $\mu_\alpha$  has stretched-exponential decay of correlations, that is, there exists  $C > 0$  and  $\tau \in (0, 1)$  such that

$$\left| \int (h_1 \circ \varphi_\alpha^n) h_2 d\mu_\alpha - \int h_1 d\mu_\alpha \cdot \int h_2 d\mu_\alpha \right| \leq C e^{-\tau \sqrt{n}} \|h_1\|_\infty \|h_2\|_\beta$$

for all large  $n$  and observables  $h_1 \in L^\infty(\mu)$  and  $h_2 \in \mathcal{H}_\beta$ ;

- (4)  $\mu_\alpha$  has stretched-exponential large deviations, meaning that there exists  $\zeta \in (0, \frac{1}{2})$  such that for all  $\delta > 0$  and  $h \in \mathcal{H}_\beta$  there exists  $\gamma > 0$  satisfying

$$\mu \left( \left| \frac{1}{n} \sum_{j=0}^{n-1} h \circ \varphi^j - \int h d\mu_\alpha \right| > \delta \right) \leq e^{-\gamma n^\zeta} \text{ for all large } n;$$

- (5)  $\mu_\alpha$  satisfies the central limit theorem, the almost sure invariance principle, the local limit theorem and the Berry-Esseen theorem for Hölder observables

Furthermore, all these properties hold for every  $\varphi$  that is  $C^3$ -close enough to  $\varphi_\alpha$ .

Our strategy to deduce the later ergodic properties is to use recent contributions to the study of stretched-exponential large deviations and limit theorems using Markov induced maps e.g. by Melbourne and Nicol [MN08], Alves, Luzzatto, Freitas, Vaienti [ALFV11] or Alves and Schnelmann [AS11]. Our next main result concerns the coexistence of a dense set points with a negative and a positive Lyapunov exponents together with a full Lebesgue measure set of points with only positive Lyapunov exponents.

**Theorem C.** Let  $\varphi_\alpha : (0, 1] \times I \rightarrow (0, 1] \times I$  be a generalized Viana map  $\varphi_\alpha(\theta, x) = (g(\theta), f_\alpha(\theta, x))$  such that  $h(x) = a_0 - x^2$  is Misiurewicz and let  $\mathcal{V}$  be a  $C^3$ -open set of generalized skew-Viana maps. Then there exists a residual subset  $\mathcal{R} \subset \mathcal{V}$  such that for every  $\varphi \in \mathcal{R}$

- (1) the map  $\varphi$  has countably many saddle points;
- (2) there is a dense set of points  $D \subset \Lambda(\varphi)$  with a negative Lyapunov exponent, that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial x} \right\| < 0 \quad \text{for all } (\theta, x) \in D;$$

- (3) all periodic points are hyperbolic ;
- (4) Lebesgue almost every point in  $\Lambda$  has two positive Lyapunov exponents.

In view of the previous theorem an interesting question is to understand if, at least generically, all Lyapunov exponents are bounded away from zero. The later goes in the direction of understanding possible phase transitions for the topological pressure  $P(t)$  of the generalized Viana-map  $\varphi_\alpha$  with respect to the family of potentials  $\psi_{\alpha,t} = -t \log |\partial_x \varphi_\alpha|$ , that is, parameters  $t$  such that  $\psi_{\alpha,t}$  has none or more than one equilibrium state. Finally, our last main result concerns the dynamics of quadratic skew-products where the parameters are driven among hyperbolic parameters. We will say that  $K \subset (0, 1] \times I_0$  is a *Cantor set of curves* if for any

$\theta \in (0, 1]$  the set  $K_\theta := K \cap (\{\theta\} \times I_0)$  is a Cantor set and the map  $\theta \mapsto K_\theta$  is continuous in the Hausdorff topology. First recall that a quadratic map  $h(x) = a_0 - x^2$  is hyperbolic if it has finitely many hyperbolic attracting periodic points and the complement of the basins of attraction is a hyperbolic set.

**Theorem D.** *Consider the skew-product  $\varphi_\alpha : (0, 1] \times I \rightarrow (0, 1] \times I$  given by  $\varphi_\alpha(\theta, x) = (g(\theta), f_\alpha(\theta, x))$  and such that  $h(x) = a_0 - x^2$  is hyperbolic. Then for every small  $\alpha > 0$  the map  $\varphi_\alpha$  has a unique SRB measure  $\nu$  supported in an attractor  $\mathcal{G}$  such that all points have one positive and one negative Lyapunov exponents. Moreover, the complement  $K$  of the basin of attraction  $B(\nu)$  is a  $\varphi_\alpha$ -invariant and expanding Cantor set of curves. Finally, the same holds for every  $\varphi$  that is  $C^3$ -close enough to  $\varphi_\alpha$*

Let us mention that the SRB measure above is supported on a topological attractor  $\mathcal{G} \subset (0, 1] \times I_0$  as described in detail in Section 7. Finally, some interesting questions are to understand if the transversality properties of admissible curves yield that the attractor  $\mathcal{G}$  has non-empty interior and the SRB measure is absolutely continuous with respect to Lebesgue (we refer the reader to [Tsu01, Vo11] where similar problems are considered). Finally, since almost every parameter is regular or stochastic for the quadratic map it would be interesting to understand whether either of Theorem A or Theorem D hold for quadratic skew-products and Lebesgue almost every parameter  $a \in (0, 2]$ .

**2.3. Some applications.** Let us finish this section with some examples.

*Example 2.2* (Viana maps). The class of maps considered in [Vi97] fit in the previous setting. In fact, assume  $d \geq 16$  and take the Markov expanding map on  $(0, 1]$  given by  $g(\theta) = d\theta - [d\theta]$  (where  $[\cdot]$  stands for the integer part). Then  $g$  admits a  $C^3$ -extension to the boundary elements and one can identify the boundary points of  $(0, 1]$  to obtain the  $C^3$  expanding map on the circle  $\mathbb{S}^1$  given by  $g(\theta) = d\theta \pmod{1}$ , thus recovering the previous setting. In particular, in this context Theorems A and B are consequences of [Vi97, Al00].

In this context, it follows from Theorems C that  $C^1$ -generic transformations in the  $C^3$  neighborhood of Viana maps exhibit coexistence of a dense set of points with one negative Lyapunov exponent while Lebesgue almost every point has only positive Lyapunov exponents. Finally, it follows from Theorem D that quadratic skew-products with parameters driven among hyperbolic ones admit a unique SRB measure, it is hyperbolic and the complement of its basin of attraction is an expanding Cantor set of lines.

In the next class of examples we present a robust class of Markov expanding maps with discontinuities and infinitely many invertibility domains.

*Example 2.3* (Quadratic skew-products over piecewise linear expanding maps). Let  $\mathcal{P}$  be an arbitrary countable partition of the unit interval  $(0, 1]$  in subintervals  $(\omega_i)_{i \in S}$  with size smaller or equal to  $\frac{1}{d}$  and let  $g_0$  be piecewise linear satisfying  $g_0(\omega_i) = (0, 1]$ . Since  $|g'_0(\theta)| \geq d$  and  $g'_0(\theta)' = 0$  for all  $\theta \in (0, 1]$  then it is clear that  $g_0$  is a piecewise Markov expanding map and satisfies the R nyi condition. See Figure 1 above. Moreover, if  $\tilde{g}$  is a Markov expanding map that is  $\varepsilon$ - $C^3$ -close enough to  $g$  then it follows from Lemma 3.1 that it also satisfies the R nyi condition thus satisfying all the hypothesis to be used as base dynamics.

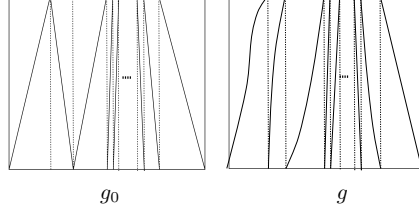


FIGURE 1. Small perturbation of piecewise linear expanding map

### 3. PRELIMINARIES

In this section we recall some definitions and preliminaries that will be used in the proof of the main results.

#### 3.1. One dimensional dynamics.

*Combinatorial description of Markov expanding maps.* Here we describe the Markov expanding maps  $g$  from the combinatorial point of view. Let  $\mathcal{P} = \{\omega_i\}_{i \in S, S \subset \mathbb{N}}$  be the Markov partition for  $g$ . Then there is a semi-conjugacy between the dynamics of  $g$  and the full shift  $\sigma : S^{\mathbb{N}} \rightarrow S^{\mathbb{N}}$  given by

$$\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, s_3, \dots),$$

where the semi-conjugation is given by the *itinerary map*  $\iota : \mathbb{N}^{\mathbb{N}} \rightarrow (0, 1]$  defined as  $\iota(s_0, s_1, s_2, \dots) = \theta$  and  $\theta$  is the only point in  $(0, 1]$  satisfying  $g^j(\theta) \in \omega_{s_j}$  for all  $j$ . Set  $\mathcal{P}^{(n)} = \bigvee_{j=0}^{n-1} g^{-j}(\mathcal{P})$  and for any partition element  $\omega \in \mathcal{P}^{(n)}$  define  $\iota_n = \iota_n(\omega) = (s_1, s_2, \dots, s_{n-1})$  as its  $n$ -th *itinerary*. For simplicity we will denote by  $\omega_{(s_0, s_1, s_2, \dots, s_{n-1})}$  the element of  $\mathcal{P}^{(n)}$  whose itinerary is  $(s_0, s_1, s_2, \dots, s_{n-1})$ . This will be helpful to give a precise description of points that visit to definite regions of the phase space.

*Rényi condition.* Here we show that Rényi condition is an open property among Markov expanding maps and relate this with the bounded distortion property.

**Lemma 3.1.** *Assume that  $g : (0, 1] \rightarrow (0, 1]$  is a  $C^3$ -Markov expanding map satisfying the Rényi condition  $|g''| \leq K|g'|^2$ . If  $\|g - \tilde{g}\|_{C^3} < \varepsilon$  for small  $\varepsilon$  then  $\tilde{g}$  satisfies  $|\tilde{g}''| \leq \tilde{K}|\tilde{g}'|^2$  with  $\tilde{K} = (d - \varepsilon)^{-2}\varepsilon + (1 - \varepsilon)^{-2}K$ .*

*Proof.* Assume that  $\|g - \tilde{g}\|_{C^3} < \varepsilon$ . Using that  $g$  is expanding it follows that  $|g' - \tilde{g}'| < \varepsilon < \varepsilon|g'|$  and consequently

$$\begin{aligned} \frac{|\tilde{g}''|}{|\tilde{g}'|^2} &\leq \frac{|g''| + |g'' - \tilde{g}''|}{|\tilde{g}'|^2} \leq \frac{1}{(d - \varepsilon)^2} \|g - \tilde{g}\|_{C^2} + \frac{|g''|}{(|g'| - |g' - \tilde{g}'|)^2} \\ &\leq \frac{1}{(d - \varepsilon)^2} \|g - \tilde{g}\|_{C^2} + \frac{1}{(1 - \varepsilon)^2} \frac{|g''|}{|g'|^2} \leq \frac{1}{(d - \varepsilon)^2} \|g - \tilde{g}\|_{C^2} + \frac{K}{(1 - \varepsilon)^2}. \end{aligned}$$

This proves that  $\tilde{g}$  also satisfies the Rényi condition and proves the lemma.  $\square$

In the second lemma we collect some bounded distortion estimates.



**Lemma 3.2.** *Let  $g$  be a  $C^3$ -Markov expanding map satisfying  $|g'(\theta)| \geq d$  and the R enyi condition  $|g''| \leq K|g'|^2$ . Then, for all  $\omega \in \mathcal{P}^{(n)}$  and  $\theta_1, \theta_2 \in \omega$*

$$\exp\left(-\frac{dK}{d-1}\right) \leq \frac{|(g^n)'(\theta_1)|}{|(g^n)'(\theta_2)|} \leq \exp\left(\frac{dK}{d-1}\right).$$

*Proof.* Let  $\theta_1, \theta_2 \in \omega$  for some  $\omega \in \mathcal{P}^{(n)}$  be given. We may assume, without loss of generality, that  $g^n|_\omega$  is increasing. Then

$$\begin{aligned} \left| \log \frac{(g^n)'(\theta_1)}{(g^n)'(\theta_2)} \right| &\leq \sum_{j=0}^{n-1} \left| \log(g'(g^j(\theta_1))) - \log(g'(g^j(\theta_2))) \right| \\ &= \sum_{j=0}^{n-1} \left| \int_{g^j(\theta_1)}^{g^j(\theta_2)} \frac{g''(\theta)}{g'(\theta)} d\theta \right| \leq K \sum_{j=0}^{n-1} \left| \int_{g^j(\theta_1)}^{g^j(\theta_2)} g'(\theta) d\theta \right| \\ &= K \sum_{j=1}^n |g^j(\theta_1) - g^j(\theta_2)| \leq K \sum_{j=1}^n d^{-(n-j)} |g^n(\theta_1) - g^n(\theta_2)| \end{aligned}$$

which is clearly bounded from above by  $K(1 - d^{-1})^{-1}$ . Since  $\theta_1, \theta_2$  we arbitrary then the lower bound also holds. This finishes the proof of the lemma.  $\square$

Finally let us recall that it is well known that if  $g$  is a Markov expanding map with the R enyi condition then there exists a unique absolutely continuous invariant probability measure. We will use the following strong Gibbs property for Lebesgue.

**Corollary 3.3.** *Let  $g$  be a  $C^3$ -Markov expanding map satisfying  $|g'(\theta)| \geq d$  and the R enyi condition  $|g''| \leq K|g'|^2$ . Then, for all  $\omega \in \mathcal{P}^{(n)}$  and  $\theta^* \in \omega$*

$$\exp\left(-\frac{dK}{d-1}\right) \leq \frac{\text{Leb}(\omega)}{|(g^n)'(\theta^*)|^{-1}} \leq \exp\left(\frac{dK}{d-1}\right)$$

*Proof.* The proof is a simple application of the usual change of coordinates to the diffeomorphism  $g^n|_\omega: \omega \rightarrow (0, 1]$  by means that

$$\int_\omega |(g^n)'(\theta)| d\theta = \text{Leb}(g^n(\omega)) = \text{Leb}(f(0, 1]) = 1$$

together with the previous bounded distortion estimates.  $\square$

*Hyperbolicity in dimension one.* In [Sm00], Smale proposed the density of hyperbolicity in dimension one as one of the problems for the 21st century. Recall that a  $C^1$  endomorphism of a compact interval is *hyperbolic* if it has finitely many hyperbolic attracting periodic points and the complement of the basins of attraction is a hyperbolic set. Such major achievement was obtained by Kozlovski, Shen and van Strien.

**Theorem 3.4.** *(Theorem 2 in [KSvS07a]) Hyperbolic maps are dense in the space of  $C^k$  maps of the compact interval or the circle for  $k = 1, 2, \dots, \infty, \omega$ .*

Here we will need to obtain hyperbolicity for the composition of an arbitrary number of interval maps. More precisely, we deduce the following proposition as a consequence of the later theorem.

**Proposition 3.5.** *Let  $C^k([a, b])$  denote the space of  $C^k$  maps on the compact interval  $[a, b]$ ,  $k = 1, 2, \dots, \infty, \omega$ . For any  $n \geq 1$  there exists an open and dense set  $\mathcal{H}_n \subset [C^k([a, b])]^n$  such that the map  $g_{\pi(n)} \circ g_{\pi(n-1)} \circ \dots \circ g_{\pi(1)} \in C^k([a, b])$  is hyperbolic for every  $(g_1, g_2, \dots, g_n) \in \mathcal{H}_n$  and every  $n$ -permutation  $\pi$ .*



*Proof.* Fix an arbitrary  $n \geq 1$  and for any permutation  $\pi$  consider the continuous transformation  $\Phi_\pi$  defined by

$$\begin{aligned} \Phi_\pi : C^k([a, b]) \times \cdots \times C^k([a, b]) &\rightarrow C^k([a, b]) \\ (g_1, g_2, \dots, g_n) &\mapsto g_{\pi(n)} \circ g_{\pi(n-1)} \circ \cdots \circ g_{\pi(1)}. \end{aligned}$$

Assume for notational simplicity, without any loss of generality, that  $\pi = id$  or equivalently  $\Phi_\pi(g_1, g_2, \dots, g_n) = g_n \circ g_{n-1} \circ \cdots \circ g_1$ . Since hyperbolicity is a robust property it follows from the previous theorem that the set  $\mathcal{A} \subset C^k([a, b])$  of hyperbolic maps is open and dense. Since  $\Phi_\pi$  is continuous this implies that the set  $\mathcal{H}_\pi = \Phi_\pi^{-1}(\mathcal{A})$  is open. Moreover, Theorem 3.4 yields that for any given  $(g_1, g_2, \dots, g_{n-1})$  there exists a dense set of maps  $g_n$  in  $C^k([a, b])$  such that  $g_n \circ g_{n-1} \circ \cdots \circ g_1$  is hyperbolic. This proves that  $\mathcal{H}_\pi$  contains a dense set in  $C^k([a, b])^n$ , thus is an open and dense set. Now we define the set  $\mathcal{H}_n = \bigcap_\pi \mathcal{H}_\pi$ . Since the intersection is over  $n!$  elements  $\pi$  it is clear that  $\mathcal{H}_n$  is an open and dense set in  $[C^k([a, b])]^n$  that satisfies the assertion of the proposition. This finishes the proof of the proposition.  $\square$

*Fibered expansion.* Now we collect some results on a mechanism to obtain expansion in the fiber direction. Roughly, expansion for random composition of perturbations of a Misiurewicz quadratic map  $h(x) = a_0 - x^2$  is obtained if orbits avoid the critical region and the loss of expansion in each return to the critical region is proportional to the return depth. More precisely,

**Proposition 3.6.** [Vi97, Lemmas 2.4 and 2.5] *Take  $h(x) = a_0 - x^2$  and for  $\alpha > 0$  consider  $f(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$ . There are constants  $0 < \kappa < 1$  and  $0 < \eta \leq \frac{1}{3}$  (depending only on  $h$ ) and  $\delta_1, C_2 > 0$ ,  $\sigma_1, \sigma_2 > 1$  such that, for every small  $\alpha > 0$  there exists  $N(\alpha) \geq 1$  satisfying:*

- (1)  $K_0 \log \frac{1}{\alpha} \leq N(\alpha) \leq K_1 \log \frac{1}{\alpha}$  for some uniform constants  $K_0, K_1 > 0$ ;
- (2) Given an interval  $I \subset I_0$ , for every  $(\theta, x) \in (0, 1] \times I$  with  $|x| < 2\sqrt{\alpha}$  the iterates  $(\theta_j, x_j) = \varphi^j(\theta, x)$  satisfy  $|x_j| \geq \sqrt{\alpha}$  for every  $j = 1 \dots N(\alpha)$ ;
- (3)  $\prod_{j=0}^{N(\alpha)-1} |\partial_x f(\theta_j, x_j)| \geq |x| \alpha^{-1+\eta}$  for all  $(\theta, x) \in (0, 1] \times I$  with  $|x| < 2\sqrt{\alpha}$ ;
- (4) For every  $(\theta, x) \in (0, 1] \times I$  with  $\sqrt{\alpha} \leq |x| < \delta_1$  there exists  $p(x) \leq N(\alpha)$  so that  $\prod_{j=0}^{p(x)-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{\kappa} \sigma_1^{p(x)}$ ;
- (5)  $\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sqrt{\alpha} \sigma_2^n$  for every  $(\theta, x) \in (0, 1] \times I$  with  $|x_j| \geq \sqrt{\alpha}$  for every  $j = 1, \dots, n-1$ ; and
- (6)  $\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \sigma_2^n$  for all  $(\theta, x) \in (0, 1] \times I$  such that  $|x_j| \geq \sqrt{\alpha}$  with  $j = 1, \dots, n-1$  and  $|x_n| \leq \delta_1$ .

Taking the previous proposition into account it is important to estimate how close typical points return close to the critical region.

**3.2. Partial hyperbolicity and admissible curves.** Under our assumptions on the generalized Viana maps of Definition 2.1, we obtain that the map is indeed partially hyperbolic in the sense that the dynamics along the horizontal direction dominates the dynamics along the vertical fibers. This will be made precise in terms of admissible curves as we now describe.

*Definition 3.7.* A curve  $\hat{Y} = \text{graph}(Y)$  with  $Y : (0, 1] \rightarrow I_0$  is an *admissible curve* if it is  $C^2$  differentiable,  $|Y'(\theta)| \leq \alpha$  and  $|Y''(\theta)| \leq \alpha$  for every  $\theta \in (0, 1]$ .

The strong expansion assumption on the Markov map  $g$  yields a domination property as we now describe.

**Lemma 3.8.** *If  $\hat{Y}$  is an admissible curve then for every  $\omega \in \mathcal{P}^{(n)}$  it follows that  $\varphi^n(\hat{Y} \mid_\omega)$  is an admissible curve. In particular  $\varphi^n(\hat{Y})$  is an at most countable collection of admissible curves.*

*Proof.* This lemma follows from [Vi97, Lemma 2.1], whose argument we reproduce here for completeness and the reader's convenience. Since it is enough to prove the lemma for  $n = 1$  and use the argument recursively, let  $\hat{Y} = \text{graph } Y$  be an admissible curve and take  $\omega \in \mathcal{P}$ . Then, for every  $\theta \in P$

$$\varphi(\hat{Y}(\theta)) = \varphi(\theta, Y(\theta)) = (g(\theta), f(\theta, Y(\theta))) = (g(\theta), Y_1(g(\theta)))$$

where, by the chain rule and definition of  $Y_1 : (0, 1] \rightarrow I$ ,

$$|Y_1'(g(\theta))| \leq \frac{1}{|g'(\theta)|} \left| \partial_\theta f(\hat{Y}(\theta)) + \partial_x f(\hat{Y}(\theta)) Y'(\theta) \right| \leq \frac{2\pi + 4}{16} \alpha < \alpha.$$

Analogously it is not hard to check that

$$\begin{aligned} |Y_1'(g(\theta))| &\leq \frac{1}{|g'(\theta)|^2} \left| \partial_{\theta\theta} f(\hat{Y}(\theta)) + \partial_{x\theta} f(\hat{Y}(\theta)) Y'(\theta) + \partial_{\theta x} f(\hat{Y}(\theta)) Y'(\theta) \right. \\ &\quad \left. + \partial_x f(\hat{Y}(\theta)) Y''(\theta) + \partial_{xx} f(\hat{Y}(\theta)) (Y'(\theta))^2 - Y_1'(g(\theta)) g''(\theta) \right|, \end{aligned}$$

which is smaller than  $\alpha$  since the partial derivatives of  $f$  are smaller compared with the term  $1/|g'|^2 \leq 1/16^2$ . This proves that  $\hat{Y}_1 = \varphi(\hat{Y} \mid_\omega)$  is an admissible curve. Since  $\mathcal{P}$  is at most countable then  $\varphi(\hat{Y})$  is the union of at most countable admissible curves. This finishes the proof of the lemma.  $\square$

The crucial property of admissible curves is that their images by  $\varphi$  are non-flat.

**Lemma 3.9.** *Let  $\hat{Y} = \text{graph } Y$  be an admissible curve and set  $\hat{Y}_1(\theta) = \varphi(\hat{Y})(\theta) = (g(\theta), Y_1(\theta))$ . Then  $|Y_1'(\theta)| \geq \alpha/2$  or  $|Y_1''(\theta)| \geq 4\alpha$  for every  $\theta \in (0, 1]$  and*

$$\text{Leb} \left( \theta \in (0, 1] : \hat{Y}_1(\theta) \in (0, 1] \times I \right) \leq \frac{6|I|}{\alpha} + 2\sqrt{\frac{|I|}{\alpha}}.$$

for any interval  $I \subset I_0$ .

*Proof.* The proof follows the same ideas of [Vi97, Lemma 2.2] even with the presence of discontinuities for  $g$ . In fact, set  $\hat{Y}(\theta) = (\theta, Y(\theta))$  and notice that for all  $\theta \in (0, 1]$

$$Y_1'(\theta) = \partial_\theta f(\hat{Y}(\theta)) + \partial_x f(\hat{Y}(\theta)) Y'(\theta) = 2\pi\alpha \cos(2\pi\theta) - 2Y(\theta) Y'(\theta) \quad (3.1)$$

and  $Y_1''(\theta) = -4\pi^2 \sin(2\pi\theta) - 2Y'(\theta)^2 - 2Y(\theta) Y''(\theta)$ . If  $\theta \in \mathcal{A} = \{\tilde{\theta} \in (0, 1] : |\sin(2\pi\tilde{\theta})| \leq \frac{1}{3}\}$  then  $|\cos(2\pi\theta)| \geq \frac{11}{12}$  and it follows from (3.1) that  $|Y_1'(\theta)| \geq (\frac{11\pi}{6} - 4)\alpha > \frac{\alpha}{2}$ . Otherwise, for  $\theta \in (0, 1] \setminus \mathcal{A}$  it follows that  $|Y_1''(\theta)| \geq (\frac{4\pi^2}{3} - 2 - 2\alpha)\alpha \geq 4\alpha$ . This proves the first statement of the lemma.

Now, notice that  $\mathcal{A}$  has three connected components and  $|Y_1'(\theta)| \geq \alpha/2$  for all  $\theta \in \mathcal{A}$ . Moreover, since the map  $\theta \mapsto Y_1(\theta)$  is  $C^1$ -differentiable then applying the Mean Value Theorem applied to  $Y_1$  on each connected component of  $\mathcal{A}$  it follows that  $\text{Leb} \left( \theta \in \mathcal{A} : \hat{Y}_1(\theta) \in (0, 1] \times I \right) \leq \frac{6|I|}{\alpha}$ . Using that  $|Y_1''| \geq 4\alpha$  on the two connected components of  $(0, 1] \setminus \mathcal{A}$  a similar argument shows that  $\text{Leb} \left( \theta \in (0, 1] \setminus \mathcal{A} : \hat{Y}_1(\theta) \in (0, 1] \times I \right) \leq 4\sqrt{\frac{|I|}{4\alpha}} = 2\sqrt{\frac{|I|}{\alpha}}$ . The proof of the lemma is now complete.  $\square$

We obtain the following very useful consequence.

**Corollary 3.10.** *Let  $\hat{Y} = \text{graph}(Y)$  be an admissible curve and set  $\hat{Y}_j = \varphi^j(\hat{Y})$ . Then there exists  $C > 0$  (depending only on  $g$ ) such that for any subinterval satisfying  $|I| \leq \alpha$  it holds*

$$\text{Leb}\left(\theta \in (0, 1] : \hat{Y}_j(\theta) \in (0, 1] \times I\right) \leq C \sqrt{\frac{|I|}{\alpha}} \quad \text{for all } j \geq 1.$$

*Proof.* First note that the case  $j = 1$  corresponds to the previous lemma. Hence, let  $j \geq 2$  be arbitrary and fixed. If  $\omega \in \mathcal{P}^{(j-1)}$  then  $g^{j-1}(\omega) = (0, 1]$  and  $\varphi^{j-1}(\hat{Y} |_\omega)$  is an admissible curve. Therefore,  $\text{Leb}(\theta \in (0, 1] : \varphi(\varphi^{j-1}(\hat{Y} |_\omega))(\theta) \in (0, 1] \times I) \leq 6\sqrt{\frac{|I|}{\alpha}}$  for any subinterval  $I$  satisfying  $|I| \leq \alpha$ . Then one can use the bounded distortion property to get

$$\begin{aligned} \text{Leb}\left(\theta \in (0, 1] : \hat{Y}_j(\theta) \in \hat{I}\right) &= \sum_{\omega \in \mathcal{P}^{(j-1)}} \text{Leb}\left(\theta \in \omega : \varphi(\varphi^{j-1}(\hat{Y} |_\omega))(\theta) \in \hat{I}\right) \\ &\leq \sum_{\omega \in \mathcal{P}^{(j-1)}} 6 \frac{Kd}{d-1} \sqrt{\frac{|I|}{\alpha}} |\omega| \leq C \sqrt{\frac{|I|}{\alpha}}, \end{aligned}$$

where  $\hat{I} = (0, 1] \times I$  and  $C = 6K > 0$  depends only on  $g$ . The proof of the corollary is now complete.  $\square$

*Remark 3.11.* Let us mention that the bound in the right hand side of the expression in Corollary 3.10 depends on  $\alpha$  and it increases when  $\alpha$  approaches zero. In fact, as  $\alpha$  will be required to be very small for a the large deviations argument in Subsection 4.2, it is necessary to obtain similar estimates where the right hand side above does not depend on  $\alpha$ .

#### 4. POSITIVE LYAPUNOV EXPONENTS FOR THE SKEW-PRODUCT $\varphi_\alpha$

In this section we study the recurrence of typical points near the critical region and deduce existence of positive Lyapunov exponents Lebesgue almost everywhere.

**4.1. Recurrence estimates.** For notational simplicity set  $\hat{J}(r) = (0, 1] \times J(r)$  for every  $r$ . In the next proposition we show that for a sufficiently large iterate  $\varphi^{M(\alpha)}$  of  $\varphi$  the measure of the set of points which have exponential deep returns decay exponentially fast with a rate that does not involve  $\alpha$ . More precisely, if  $0 < \eta < \frac{1}{3}$  is as given in Proposition 3.6 we obtain the following.

**Proposition 4.1.** *There exists  $\varepsilon > 0$  small so that every  $\varphi$  that is  $\varepsilon$ - $C^3$ -close to  $\varphi_\alpha$  satisfies the following property: there exists  $\tilde{C}, \beta > 0$  and for any given  $\alpha > 0$  there is a positive integer  $M = M(\alpha) < N(\alpha)$  such that if  $\hat{Y} = \text{graph}(Y)$  is an admissible curve, then*

$$\text{Leb}\left(\theta \in (0, 1] : \hat{Y}_M(\theta) \in \hat{J}(r-2)\right) \leq \tilde{C}e^{-5\beta r}$$

for every  $r \geq (\frac{1}{2} - 2\eta) \log \frac{1}{\alpha}$ .

*Proof.* Let  $\hat{Y}$  be a fixed arbitrary admissible curve and set  $\hat{Y}_j(\theta) = \varphi_\alpha^j(\hat{Y}(\theta)) = (g^j(\theta), Y_j(\theta))$ . On the one hand using Corollary 3.10 we deduce that

$$\text{Leb}\left(\theta \in (0, 1] : \hat{Y}_j(\theta) \in \hat{J}(r-2)\right) \leq C\alpha^{-\frac{1}{2}} \sqrt{|J(r-2)|} \leq C\alpha^{-\frac{1}{4}} e^{-\frac{1}{2}(r-2)},$$

which satisfies the assertion in the corollary with  $\tilde{C} = Ce$  and  $\beta = \frac{1}{15}$ , provided that  $\frac{r}{6} \geq \frac{1}{4} \log \frac{1}{\alpha}$ . So, through the remaining we assume  $(\frac{1}{2} - 2\eta) \log \frac{1}{\alpha} \leq r \leq \frac{3}{2} \log \frac{1}{\alpha}$ . Let  $M = M(\alpha)$  be maximal such that  $32^M \alpha \leq 1$ , and note that  $M < N$ . One can write

$$\begin{aligned} \text{Leb} \left( \theta \in (0, 1] : \hat{Y}_M(\theta) \in \hat{J}(r-2) \right) &= \sum_{\omega \in \mathcal{P}^{(M)}} \text{Leb} \left( \theta \in \omega : \hat{Y}_M(\theta) \in \hat{J}(r-2) \right) \\ &= \sum_{\underline{s} \in S^M} \text{Leb} \left( \theta \in \omega_{\underline{s}} : \hat{Y}_M(\theta) \in \hat{J}(r-2) \right), \end{aligned}$$

where  $\underline{s} = (s_1, s_2, \dots, s_M) \in S^M$  is the itinerary for the elements of  $\mathcal{P}^{(M)}$ . The strategy is to subdivide itineraries  $\underline{s} = (s_1, s_2, \dots, s_M) \in S^M$  according to its average depth  $\sum_{i=1}^M s_i(\theta)$ . On the other hand, we use a large deviations argument to show that points with large average depth decrease exponentially fast. On the other hand, admissible curves associated to points with smaller average have vertical displacement.

*Claim:* For any  $L \geq 1$  there exists  $\zeta \in (0, 1)$  and for any admissible curve  $\hat{Z}$  there are disjoint collections  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of elements of  $\mathcal{P}$  such that the image  $\hat{Z}_1(\theta) = \varphi(\hat{Z})(\theta) = (g(\theta), Z_1(\theta))$  satisfies  $|Z_1|_{\omega} - Z_1|_{\tilde{\omega}}| \geq \frac{\alpha}{100}$  for all  $\omega \in \mathcal{P}_1$  and  $\tilde{\omega} \in \mathcal{P}_2$  and

$$\zeta \leq \text{Leb} \left( \bigcup_{\omega \in \mathcal{P}_1} \omega \right) \leq \text{Leb} \left( \bigcup_{\omega \in \mathcal{P}_2} \omega \right) \leq 1 - \zeta.$$

*Proof of the Claim:* It follows from Lemma 3.9 that the admissible curve  $\hat{Z}_1$  has two critical points  $\theta'_1 < \theta'_2$  one in each connected component of the set  $(0, 1] \setminus \mathcal{A} = \{\theta \in (0, 1] : |\sin(2\pi\theta)| > \frac{1}{3}\}$ . Hence  $\frac{1}{2\pi} \arcsin(\frac{1}{3}) < \theta'_1 < \frac{1}{2} - \frac{1}{2\pi} \arcsin(\frac{1}{3})$  and also  $\frac{1}{2} + \frac{1}{2\pi} \arcsin(\frac{1}{3}) < \theta'_2 < 1 - \frac{1}{2\pi} \arcsin(\frac{1}{3})$ .

On the one hand, if  $\theta'_1, \theta'_2 \notin [\frac{1}{4}, \frac{3}{4}]$  then  $[\theta'_1, \theta'_1 + \frac{1}{16}]$  and  $(\theta'_2 - \frac{1}{16}, \theta'_2]$  are disjoint intervals that do not intersect the middle component  $[\frac{1}{2} - \frac{1}{2\pi} \arcsin(\frac{1}{3}), \frac{1}{2} + \frac{1}{2\pi} \arcsin(\frac{1}{3})]$  in  $\mathcal{A}$ . Then, using that  $\hat{Z}_1|_{[\theta'_1, \theta'_2]}$  is strictly monotone and  $|Z'_1|_{\mathcal{A}}| \geq \frac{\alpha}{2}$

$$\inf Z_1|_{[\theta'_1, \theta'_1 + \frac{1}{16}]} - \sup Z_1|_{[\theta'_2 - \frac{1}{16}, \theta'_2]} \geq \frac{\alpha}{2\pi} \arcsin(\frac{1}{3}) \geq \frac{\alpha}{100}.$$

On the other hand, if  $\theta'_1 \geq \frac{1}{4}$  then  $(0, \frac{1}{16}]$  and  $(\theta'_1 - \frac{1}{16}, \theta'_1]$  are disjoint intervals. Moreover, using that  $Z_1|_{(0, \theta'_1]}$  is strictly monotone and that  $|Z'_1|_{(\frac{1}{16}, \frac{3}{16})}| \geq 4\alpha$  we obtain analogously  $\inf Z_1|_{(\theta'_1 - \frac{1}{16}, \theta'_1]} - \sup Z_1|_{(0, \frac{1}{16}]} \geq \frac{4\alpha}{64} \geq \frac{\alpha}{100}$ . A similar reasoning holds for the case  $\theta'_2 \leq \frac{3}{4}$ .

Since all partition elements of  $\mathcal{P}$  have length smaller or equal to  $\frac{1}{16}$ , there are collections  $\mathcal{P}'_1$  and  $\mathcal{P}'_2$  of elements in  $\mathcal{P}$  such that  $|Z_1|_{\omega} - Z_1|_{\tilde{\omega}}| \geq \frac{\alpha}{100}$  for all  $\omega \in \mathcal{P}'_1$  and  $\tilde{\omega} \in \mathcal{P}'_2$ . In addition, using that  $\mathcal{P}'_1$  and  $\mathcal{P}'_2$  are disjoint we get

$$1 - \frac{1}{16} \geq \text{Leb} \left( \bigcup_{\omega \in \mathcal{P}'_i} \omega \right) \geq \frac{1}{16}$$

for  $i = 1, 2$  and we set  $\zeta = \frac{1}{16}$ . Finally if  $\text{Leb}(\bigcup_{\omega \in \mathcal{P}_1} \omega) \leq \text{Leb}(\bigcup_{\omega \in \mathcal{P}_2} \omega)$  we define  $(\mathcal{P}_1, \mathcal{P}_2) := (\mathcal{P}'_1, \mathcal{P}'_2)$  while otherwise  $(\mathcal{P}_1, \mathcal{P}_2) := (\mathcal{P}'_2, \mathcal{P}'_1)$ . This finishes the proof of our claim.  $\square$

Let  $\omega, \tilde{\omega} \in \mathcal{P}^{(M)}$  be arbitrary with  $\iota_M(\omega) = (s_1, \dots, s_M)$  and  $\iota_M(\tilde{\omega}) = (\tilde{s}_1, \dots, \tilde{s}_M)$ , and assume that  $|Y_M(\theta)| < \sqrt{\alpha}$  for some  $\theta$  since otherwise there is nothing to prove. We collect some facts whose proof can be found in [Vi97, p.72-73]:

- (1) (Expansion estimates) Given an arbitrary  $\hat{y} \in \hat{Y}$  and  $0 \leq j \leq M-1$

$$\lambda_j := |\partial_x f^{M-j}(\varphi_\alpha^j(\hat{y}))| \geq C_2 \sigma_2^{M-j}$$

- (2) (Bounded distortion) For all  $0 \leq j \leq M-1$ , all  $(\theta_j, x_j) \in \hat{Y}_j$  and  $1 \leq i \leq M-j$  it holds that

$$\frac{1}{2} \frac{\lambda_j}{\lambda_{i+j}} \leq |\partial_x f^i(\theta_j, x_j)| \leq 2 \frac{\lambda_j}{\lambda_{i+j}}$$

- (3) (Positive frequency) If  $K = 400e^2$ ,  $t_1 = 1$  and define recursively  $t_{i+1} = \min\{t_i < s \leq M : \lambda_{t_i} \geq 2K\lambda_s\}$  then there exists  $\gamma_1 > 0$  (depending only on  $\eta$ ) such that  $k(r) = \max\{i : \lambda_{t_i} \geq 2\alpha^{-\frac{1}{2}} K e^{-r}\}$  satisfies  $k(r) \geq \gamma_1 r$ .

- (4) (Vertical displacement) For any  $1 \leq i \leq k$  and  $(s_1, \dots, s_{t_i-1})$  there are collections  $\mathcal{P}_{1,i}$  and  $\mathcal{P}_{2,i}$  as in Claim 1 (with corresponding symbols  $s_{t_i}^1, s_{t_i}^2$ ) for which the admissible curves  $\varphi^{t_i}(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^1})$  and  $\varphi^{t_i}(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^2})$  satisfy

$$|\varphi^{t_i}(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^1})(\theta) - \varphi^{t_i}(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^2})(\theta)| \geq \frac{\alpha}{100} \quad \text{for all } \theta \in (0, 1]$$

and, consequently, for any  $(s_{t_i} + 1, \dots, s_M) \in S^{M-t_i}$

$$|\varphi^M(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^1, s_{t_i}+1, \dots, s_M})(\theta) - \varphi^M(\omega_{s_1, \dots, s_{t_i-1}, s_{t_i}^2, s_{t_i}+1, \dots, s_M})(\theta)| \geq 4\sqrt{\alpha}e^{-(r-2)}.$$

Now we are in a position to finish the proof of Proposition 4.1. By a small abuse of notation, we write  $s_{t_i}(\theta) \in \mathcal{P}_{1,i}$  meaning that  $\omega_{s_{t_i}}(\theta) \in \mathcal{P}_{1,i}$ , and analogously for the collection  $\mathcal{P}_{2,i}$ . In fact, one can combine Claim 1 with property (4) above to show that  $\text{Leb}(\theta \in (0, 1] : \hat{Y}_M(\theta) \notin \hat{J}(r-2))$  is given by

$$\begin{aligned} & \sum_{s_1 \in S} \sum_{(s_2, \dots, s_{M-1}) \in S^{M-1}} \text{Leb}(\theta \in \omega_{s_1, \dots, s_M} : \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \\ & \geq \text{Leb}(\theta : s_1(\theta) \in \mathcal{P}_{1,1}) + \text{Leb}(\theta : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } \hat{Y}_M(\theta) \notin \hat{J}(r-2)), \end{aligned}$$

since admissible segments over  $\mathcal{P}_{1,1}$  and  $\mathcal{P}_{2,1}$  correspond to vertically displaced admissible curves when mapped by  $\varphi^M$  and  $\mu(\bigcup_{\omega \in \mathcal{P}_{1,1}} \omega) \leq \mu(\bigcup_{\omega \in \mathcal{P}_{2,1}} \omega)$ . Analogously, the second term in the right hand-side above satisfies

$$\begin{aligned} & \text{Leb}(\theta \in (0, 1] : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \\ & = \sum_{s_1 \notin \mathcal{P}_{1,1}} \sum_{s_2, \dots, s_{t_2-1}} \sum_{s_{t_2} \notin \mathcal{P}_{1,2}} \sum_{s_{t_2+1}, \dots, s_M} \text{Leb}(\theta : \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \\ & + \sum_{s_1 \notin \mathcal{P}_{1,1}} \sum_{s_2, \dots, s_{t_2-1}} \sum_{s_{t_2} \in \mathcal{P}_{1,2}} \sum_{s_{t_2+1}, \dots, s_M} \text{Leb}(\theta : \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \\ & \geq \sum_{s_1 \notin \mathcal{P}_{1,1}} \sum_{s_{t_2} \notin \mathcal{P}_{1,2}} \text{Leb}(\theta : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } s_{t_2}(\theta) \notin \mathcal{P}_{1,2}) \\ & + \text{Leb}(\theta : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } s_{t_2}(\theta) \in \mathcal{P}_{1,2} \text{ and } \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \end{aligned}$$

Proceeding recursively we obtain that

$$\begin{aligned}
& \text{Leb}(\theta : \hat{Y}_M(\theta) \notin \hat{J}(r-2)) \\
& \geq \text{Leb}(\theta : s_1(\theta) \in \mathcal{P}_{1,1}) + \text{Leb}(\theta : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } s_{t_2}(\theta) \in \mathcal{P}_{1,2}) \\
& \quad + \text{Leb}(\theta : s_1(\theta) \notin \mathcal{P}_{1,1} \text{ and } s_{t_2}(\theta) \notin \mathcal{P}_{1,2} \text{ and } s_{t_3}(\theta) \in \mathcal{P}_{1,3}) \\
& \quad + \dots \\
& \quad + \text{Leb}(\theta : s_{t_i}(\theta) \notin \mathcal{P}_{1,i}, \forall 1 \leq i \leq k(r) - 1 \text{ and } s_{t_{k(r)}}(\theta) \in \mathcal{P}_{1,k}),
\end{aligned}$$

proving  $\text{Leb}(\theta \in (0, 1] : \hat{Y}_M(\theta) \in \hat{J}(r-2)) \leq \text{Leb}(\theta : s_{t_i}(\theta) \notin \mathcal{P}_{1,i}, \forall 1 \leq i \leq k(r))$ . Hence, to finish the proof it is enough to prove that the previous right hand side decreases exponentially fast on  $r$ . Since  $\varphi(\theta, x) = (\tilde{g}(\theta), \tilde{f}(\theta, x))$  where  $\|g - \tilde{g}\|_{C^3} < \varepsilon$  and  $g$  is piecewise linear satisfying  $|g'| \geq d$  then it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned}
\frac{\text{Leb}(\omega_{(s_1, s_2, \dots, s_{k+\ell})})}{\text{Leb}(\omega_{(s_1, s_2, \dots, s_k)})} & \leq \frac{\text{Leb}(\tilde{g}^k(\omega_{(s_1, s_2, \dots, s_{k+\ell})}))}{\text{Leb}(\tilde{g}^k(\omega_{(s_1, s_2, \dots, s_k)}))} \exp\left(\frac{d\varepsilon}{(d-1)(d-\varepsilon)^2}\right)^2 \\
& = \text{Leb}(\omega_{(s_{k+1}, s_{k+2}, \dots, s_{k+\ell})}) \exp\left(\frac{d\varepsilon}{(d-1)(d-\varepsilon)^2}\right)^2
\end{aligned}$$

and consequently

$$\text{Leb}(\omega_{(s_1, \dots, s_{k+\ell})}) \leq \text{Leb}(\omega_{(s_1, \dots, s_k)}) \text{Leb}(\omega_{(s_{k+1}, \dots, s_{k+\ell})}) \exp\left(\frac{d\varepsilon}{(d-1)(d-\varepsilon)^2}\right)^2$$

for any  $k + \ell \geq 1$  and sequence  $(s_1, s_2, \dots, s_{k+\ell}) \in S^{k+\ell}$ . Thus if  $\varepsilon$  is small, the previous estimates together with  $k(r) \geq \gamma_1 r$  yield that

$$\begin{aligned}
\text{Leb}(\theta : s_{t_i}(\theta) \notin \mathcal{P}_{1,i}, \forall 1 \leq i \leq k(r)) & = \sum_{\{(s_1, s_2, \dots, s_M) : s_{t_i} \notin \mathcal{P}_{1,i}\}} \text{Leb}(\omega_{(s_1, s_2, \dots, s_M)}) \\
& \leq \exp\left(\frac{d\varepsilon}{(d-1)(d-\varepsilon)^2}\right)^{2k(r)} \prod_{j=1}^{k(r)} [1 - \text{Leb}(\mathcal{P}_{1,i})] \\
& \leq \left[ \exp\left(\frac{2d\varepsilon}{(d-1)(d-\varepsilon)^2}\right) (1 - \zeta) \right]^{k(r)} \\
& \leq \left[ \exp\left(\frac{2d\varepsilon}{(d-1)(d-\varepsilon)^2}\right) (1 - \zeta) \right]^{\gamma_1 r}.
\end{aligned}$$

This proves that  $\text{Leb}(\theta \in (0, 1] : \hat{Y}_M(\theta) \in \hat{J}(r-2))$  decreases exponentially fast in  $r$  provided that  $\varepsilon > 0$  is small, and finishes the proof of the proposition.  $\square$

Observe that a simpler argument would lead the same result provided the measure  $\mu$  to satisfy a strong independence property, namely, if it is Bernoulli.

**4.2. Positive Lyapunov exponents.** We are now in a position to prove Theorem A similarly to [Vi97]. First we consider the skew-product  $\varphi_\alpha$ . Let  $\gamma \in (0, 1)$  be arbitrary and fixed. For any integer  $n \geq 1$  set  $m = \lfloor \sqrt{n} \rfloor$  and  $\ell = m - M$ , where  $\lfloor \cdot \rfloor$  stands as before for the integer part. Given an admissible curve  $\hat{Y}$  and  $v \in \mathbb{R}^2$  non-colinear with  $\partial/\partial x$  there is  $C > 0$  so that  $\|D\varphi_\alpha^n(\hat{Y}(\theta))v\| \geq C|(g^n)'(\theta)| \geq Cd^n$

grows exponentially fast. Hence, it remains to estimate the derivative

$$\left\| D\varphi_\alpha^n(\hat{Y}(\theta)) \frac{\partial}{\partial x} \right\| = \prod_{j=0}^{n-1} \left| \frac{\partial f}{\partial x}(\hat{Y}_j(\theta)) \right|,$$

where the later product can be estimated according to the returns near the critical region using Proposition 3.6. We will say that  $1 \leq \nu \leq n$  is a *deep return* for  $\theta$  if  $\theta \in \omega$  for some partition element  $\omega \in \mathcal{P}^{(\nu+\ell)}$  satisfying  $\varphi^\nu(\hat{Y}|_\omega) \cap ((0, 1] \times J(m)) \neq \emptyset$ . We will say that  $1 \leq \nu \leq n$  is a *regular return* for  $\theta$  if  $\theta \in \omega$  where  $\omega \in \mathcal{P}^{(\nu+\ell)}$  satisfies  $\varphi^\nu(\hat{Y}|_\omega) \cap ((0, 1] \times J(0)) \neq \emptyset$  and  $\varphi^\nu(\hat{Y}|_\omega) \cap ((0, 1] \times J(m)) = \emptyset$ . In this case we set the *return depth*  $r_\nu(\theta) = \min\{r < m : \varphi^\nu(\hat{Y}|_\omega) \cap ((0, 1] \times J(r)) \neq \emptyset\}$ . Observe that the function  $r_\nu(\cdot)$  is constant on the elements of the partition  $\mathcal{P}^{(\nu+\ell)}$ . Moreover, since  $\varphi^\nu(\hat{Y}|_\omega)$  is a curve with slope smaller or equal to  $\alpha$  and horizontal length smaller or equal to  $16^{-\frac{\sqrt{n}}{2}}$ , if  $\nu$  is a deep return then  $\varphi^\nu(\hat{Y}|_\omega) \subset ((0, 1] \times J(m-1))$ . In consequence, from Corollary 3.10 we get

$$\begin{aligned} \text{Leb}(\theta \in (0, 1] : \exists 1 \leq \nu \leq n \text{ deep return for } \theta) \\ \leq n \text{ Leb}(\theta \in (0, 1] : \hat{Y}_\nu(\theta) \in ((0, 1] \times J(m-1))) \\ \leq n C \alpha^{-\frac{1}{4}} e^{-\frac{m}{2}} \\ \leq \alpha^{-\frac{1}{4}} e^{-\frac{1}{3}\sqrt{n}} \end{aligned}$$

for all large  $n$ . In addition, if  $\theta \in (0, 1]$  has no deep returns and  $1 \leq \nu_1 < \nu_2 < \dots < \nu_s \leq n$  are the regular returns for  $\theta$  with return depths  $r_1, \dots, r_s$  respectively, then it follows from the estimates in [Vi97, p. 76] that for all large  $n$

$$\log \left\| D\varphi_\alpha^n(\hat{Y}(\theta)) \frac{\partial}{\partial x} \right\| \geq 2cn - \sum_{i \in G_\theta} r_i(\theta), \quad (4.1)$$

where  $c = \frac{1}{3} \min\{\gamma_2, \log \sigma_2\} > 0$  and  $G_\theta = \{1 \leq i \leq s : r_{\nu_i}(\theta) \geq (\frac{1}{2} - \eta) \log \frac{1}{\alpha}\}$ . Therefore, if  $G_\theta(q) = \{i : \nu_i \equiv q \pmod{m}\}$  (for  $0 \leq q < m$ ) we obtain that

$$\begin{aligned} \text{Leb} \left( \theta \in (0, 1] : \sum_{i \in G_\theta} r_i(\theta) \geq cn \right) &\leq \sqrt{n} \text{Leb} \left( \theta \in (0, 1] : \sum_{i \in G_\theta(q)} r_i(\theta) \geq \frac{cn}{m} \right) \\ &= \sqrt{n} \sum_{R \geq \frac{cn}{m}} \sum_{\substack{(\rho_1, \dots, \rho_\tau) \\ \sum \rho_j = R}} \text{Leb}(\theta : r_{\nu_j}(\theta) = \rho_j, \forall j) \\ &\leq \sqrt{n} \sum_{R \geq \frac{cn}{m}} \binom{R + \tau}{\tau} \tilde{C}^\tau e^{-5\beta \sum_j \rho_j}, \end{aligned}$$

where  $\tau$  denotes the number of nonzero depths  $r_j$ . Using that  $R \geq \tau(\frac{1}{2} - \eta) \log \frac{1}{\alpha}$  we can take  $\alpha$  small so that  $\frac{(R+\tau)!}{R! \tau!} \tilde{C}^\tau \leq e^{\beta R}$  and since

$$\text{Leb} \left( \theta \in (0, 1] : \sum_{i \in G_\theta} r_i(\theta) \geq cn \right) \leq \sqrt{n} \sum_{R \geq c \frac{n}{m}} e^{-4\beta R} \leq e^{-\beta \sqrt{n}}$$

for all large  $n$ , it is summable. Borel-Cantelli lemma yields that for Lebesgue almost every  $\theta$  the expression  $\sum_{i \in G_\theta} r_i(\theta) \leq cn$  holds for all but finitely many values of



$n$ . Together with equation (4.1) above and using that  $\hat{Y}$  was chosen arbitrary, this proves that  $\varphi_\alpha$  has only positive Lyapunov exponents, that is,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi_\alpha^n(\theta, x)v\| \geq c$$

for Lebesgue almost every  $(\theta, x)$  and every  $v \in \mathbb{R}^2$ , proving the assertion of Theorem A for the skew-product  $\varphi_\alpha$ . Now, let  $\varphi$  be  $\varepsilon$ - $C^3$ -close to  $\varphi_\alpha$ . For  $\varphi$  let the critical region  $\mathcal{C}$  be defined by  $(\theta, x) \in \mathcal{C}$  if and only  $\det D\varphi_\alpha(\theta, x) = 0$ . It is not hard to deduce from the Implicit Function Theorem that  $\mathcal{C}$  is a  $C^2$ -smooth curve on each invertibility domain, that is, there exists a function  $\eta : (0, 1] \rightarrow I_0$  that is  $C^2$ -close to zero on each interval  $(\theta_{i+1}, \theta_i]$  satisfying  $\mathcal{C} = \text{graph}(\eta)$ . Thus one can make a  $C^2$  change of coordinates and assume that the critical region  $\mathcal{C}$  coincides with the segment  $\{x = 0\}$ . Moreover, since  $\partial_x f(\theta x) = 2x$  we may assume  $\partial_x \tilde{f}(\theta, x) = |x|\psi(\theta, x)$  with  $\psi$  close to 2, behaves like a power of the distance to the critical region. This allows to reproduce the previous argument and to show that  $\varphi$  has two positive Lyapunov exponents and finishes the proof of Theorem A.

## 5. SRB MEASURES AND THEIR STATISTICAL PROPERTIES

This section is devoted to the study of ergodic properties of these robust nonuniformly expanding transformations. In fact we show that there is a unique SRB measure and prove that it has good statistical properties. Throughout let  $\varphi$  be  $C^3$ -close to  $\varphi_\alpha$  and let  $\Lambda$  denote the corresponding attractor. We will say that  $\varphi$  is *topologically exact* if for any open set  $U$  there exists  $N = N(U) \geq 1$  such that  $\varphi^N(U) = \Lambda$ . We say that  $\varphi$  is *ergodic* (with respect to Lebesgue) if all  $f$ -invariant measurable sets are zero or full Lebesgue measure sets.

**Proposition 5.1.** *The map  $\varphi$  is topologically exact and ergodic with respect to Leb.*

*Proof.* Since the proof follows closely [AV02, Theorem C] we will omit the details.  $\square$

At this point one could use [ArS11] to obtain the existence of the absolutely continuous invariant probability measure. However, to deduce the good statistical properties in Theorem B and to deal with the critical set  $\mathcal{C}$  and discontinuities  $\mathcal{D}$  for  $\varphi$  (formed by countable vertical segments) we need to estimate the tail of the hyperbolic times c.f. [Al00, Definition 2.5]. Since the arguments follow some now standard arguments we focus on the main ingredients.

Using that  $\varphi$  is  $C^3$ -close to  $\varphi_\alpha$ , we may assume that the critical region  $\mathcal{C} = \{(\theta, x) \in (0, 1] \times I_0 : \Xi(\theta, x) = 0\}$  coincides with the segment  $\{x = 0\}$  and that  $\varphi$  behaves like a power of a distance to the critical region along the invariant vertical foliation: there exists  $B \geq 1, \beta > 0$  so that for all  $(\theta, x) \in (0, 1] \times I_0$  and all  $v \in \mathbb{R}^2$

$$(C1) \quad \frac{1}{B} \text{dist}((\theta, x), \mathcal{C}) \leq \frac{\|D\varphi(\theta, x)v\|}{\|v\|}$$

and for all points with  $\text{dist}((\theta_1, x_1), (\theta_2, x_2)) < \text{dist}((\theta_1, x_1), \mathcal{C})/2$

$$(C2) \quad \left| \log \|D\varphi(\theta_1, x_1)^{-1}\| - \log \|D\varphi(\theta_2, x_2)^{-1}\| \right| \leq B \frac{\text{dist}((\theta_1, x_1), (\theta_2, x_2))}{\text{dist}((\theta_1, x_1), \mathcal{C})^\beta}$$

$$(C3) \quad \left| \log |\det D\varphi(\theta_1, x_1)| - \log |\det D\varphi(\theta_2, x_2)| \right| \leq B \frac{\text{dist}((\theta_1, x_1), (\theta_2, x_2))}{\text{dist}((\theta_1, x_1), \mathcal{C})^\beta}.$$

This will be used to control recurrence to the critical region  $\mathcal{C}$ . A first step to deduce stretched-exponential decay of correlations using the machinery developed

in [You98, ALP05, Gou06] we need to obtain non-uniform expansion together with slow recurrence condition to both the critical region  $\mathcal{C}$ . Notice that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(\theta, x))^{-1}\|^{-1} = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial x} \right\| \geq c > 0 \quad (\text{NUE})$$

and

$$(\forall \varepsilon > 0) (\exists \delta > 0) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(\varphi^j(\theta, x), \mathcal{C}) < \varepsilon \quad (\text{SR})$$

holds for Lebesgue almost every  $(\theta, x)$ , where  $\text{dist}_\delta(z, \mathcal{C}) = \text{dist}(z, \mathcal{C})$  if  $\text{dist}(z, \mathcal{C}) < \delta$  and  $\text{dist}_\delta(z, \mathcal{C}) = 0$  otherwise. In fact, on the one hand

$$\begin{aligned} \mathcal{E}_v(\theta, x) &:= \min \left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(\theta, x))^{-1}\|^{-1} \geq c, \text{ for all } n \geq N \right\} \\ &\leq \min \left\{ N \geq 1 : r_j(\theta, x) \leq cn \ (\forall 1 \leq j \leq n) \text{ and } \sum_{j=0}^{n-1} r_j(\theta, x) \leq cn \ (\forall n \geq N) \right\}, \end{aligned}$$

is well defined and finite for Lebesgue almost every point. Furthermore, it follows that  $\text{Leb}((\theta, x) \in (0, 1] \times I_0 : \mathcal{E}_v(\theta, x) \geq n) \leq Ce^{-\gamma\sqrt{n}}$  for all large  $n$ . On the other hand, given  $\varepsilon, \delta > 0$  consider the Lebesgue almost everywhere well defined function

$$\mathcal{R}_{v, \varepsilon, \delta}(\theta, x) := \min \left\{ N \geq 1 : \sum_{j=0}^{n-1} -\log \text{dist}_\delta(\varphi^j(\theta, x), \mathcal{C}) < \varepsilon n \text{ for all } n \geq N \right\}.$$

Notice that if  $\delta = (\frac{1}{2} - 2\eta) \log \frac{1}{\alpha}$  then  $\sum_{j=0}^{n-1} -\log \text{dist}_\delta(\varphi^j(\theta, x), \mathcal{C}) \leq \sum_{j=0}^{n-1} r_j(\theta, x)$  for all  $(\theta, x)$ . Hence, the same large deviations argument of Section 4 yield that there exists  $\gamma(\varepsilon) > 0$  such that

$$\text{Leb}((\theta, x) : \mathcal{R}_{v, \varepsilon, \delta}(\theta, x) \geq n) \leq \text{Leb} \left( (\theta, x) : \sum_{j=0}^{n-1} r_j(\theta, x) > \varepsilon n \right) \leq Ce^{-\gamma(\varepsilon)\sqrt{n}}$$

for all large  $n$ . In consequence, this proves that for any  $\varepsilon, \delta > 0$  there exists  $\tilde{\gamma}(\varepsilon) = \min\{\gamma, \gamma(\varepsilon)\}$  such that

$$\text{Leb}((\theta, x) \in (0, 1] \times I_0 : \mathcal{E}_v(\theta, x) \geq n \text{ or } \mathcal{R}_{v, \varepsilon, \delta}(\theta, x) \geq n) \leq Ce^{-\tilde{\gamma}(\varepsilon)\sqrt{n}}$$

for all large  $n$ . On the one hand, despite the discontinuities, it follows from the Markov assumption and bounded distortion Lemma 3.2 that for any partition element  $\omega \in \mathcal{P}^n$  the map  $g^n|_\omega : \omega \rightarrow (0, 1]$  has bounded distortion and the backward contraction property  $d(g^{n-j}(y), g^{n-j}(z)) \leq (d - \varepsilon)^{-j} d(g^n(y), g^n(z))$  for all  $0 \leq j \leq n$  and  $y, z \in \omega$ . On the other hand, if  $n$  is a hyperbolic time for  $(\theta, x)$  as in [Al00] then there exists a neighborhood  $B_v(\theta, x) \subset \{\theta\} \times I_0$  such that  $\varphi^n|_{\omega_n \times B_v(\theta, x)}$  is a diffeomorphism onto its image that contains a ball of definite size, with bounded distortion and backward contraction property, where  $\omega_n$  denotes the element of the partition  $\mathcal{P}^{(n)}$  containing  $\theta$ . Thus, it follows from [Gou06] that there exists a  $\varphi$ -invariant absolutely continuous probability measure  $\mu$  with stretched-exponential decay of correlations. In consequence,

**Proposition 5.2.** *There exists a unique SRB measure  $\mu$  for  $\varphi$ . Moreover, the basin of attraction  $B(\mu)$  contains Lebesgue almost every point in  $\Lambda$ .*

*Proof.* Let  $\mu$  be the  $\varphi$ -invariant and ergodic probability measure constructed above. Since  $\mu \ll \text{Leb}$  then it is clearly an SRB measure because its basin of attraction

$$B(\mu) = \left\{ (\theta, x) \in \Lambda : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\varphi^j(\theta, x)} \rightarrow \mu \right\}$$

is an  $\varphi$ -invariant set and a  $\mu$ -full measure set. Using that  $\varphi$  is exact then it follows that  $B(\mu)$  has full Lebesgue measure set in  $\Lambda$ . This also proves uniqueness of the SRB measure and finishes the proof of the proposition.  $\square$

Finally, it follows from [ALFV11] that stretched-exponential decay of correlations imply on stretched-exponential large-deviation bounds and also in the central limit theorem, almost sure invariance principle, local limit theorem and Berry-Esseen theorem. This finishes the proof of Theorem B.

## 6. COEXISTENCE OF POSITIVE AND NEGATIVE LYAPUNOV EXPONENTS FOR GENERIC GENERALIZED VIANA-MAPS

This section is devoted to the proof of Theorem C where, in particular, we prove that generic generalized Viana maps have a dense set of points with negative central Lyapunov exponent. We will make use of the density of hyperbolicity for maps of the interval and Proposition 3.5.

*Proof of Theorem C.* Let  $\mathcal{V}$  be an open set of generalized Viana maps. Since every  $\varphi \in \mathcal{V}$  is conjugated to a skew-product over a (topological) Markov expanding map  $g$ . Since  $g$  is a Markov expanding map of the interval then it has a dense and countable set  $(p_k)_k$  of expanding periodic points. Moreover, since for any perturbation of a generalized Viana map the base dynamics is topologically conjugated to  $g$ , we shall assume in what follows and without loss of generality that  $g$  is fixed.

Let  $k \geq 1$  be arbitrary and let  $\pi(p_k)$  denote the period of the periodic point  $p_k$  for  $g$ . Moreover, the map  $\varphi^{\pi(p_k)}$  preserves the fibers over the fixed points  $\{p_k, \dots, g^{\pi(p_k)-1}(p_k)\}$  for  $g^{\pi(p_k)}$ . In other words, we consider the well defined the multimodal maps  $f_{i,k} : I_0 \rightarrow I_0$  (depending on  $\varphi$ ) given by

$$f_{0,k}(x) = f(g^{\pi(p_k)-1}(p_k), \cdot) \circ \dots \circ f(g(p_k), \cdot) \circ f(p_k, x),$$

and

$$f_{i,k}(x) = f(g^{i-1}(p_k), \cdot) \circ \dots \circ f(p_k, x) \circ f(g^{\pi(p_k)-1}(p_k), \cdot) \circ \dots \circ f(g^i(p_k), x),$$

for  $1 \leq i \leq \pi(p_k) - 1$ . Now it is not hard to check that one can apply Proposition 3.5 to deduce that there exists an open and dense set  $\mathcal{A}_k \subset \mathcal{V}$  such that for any  $\varphi \in \mathcal{A}_k$  the corresponding multimodal maps  $f_{i,k}$  are hyperbolic, for every  $0 \leq i \leq \pi(p_k) - 1$ . In consequence, the basins of attraction associated to the hyperbolic periodic attractors is a full Lebesgue measure, open and dense set for each of these multimodal maps. In consequence,  $\varphi$  has periodic saddles of period  $\pi(p_k)$  in the union of the fibers over the orbit of  $p_k$  by  $g$ , and there exists an open and dense set of points in the fibers with a negative Lyapunov exponent. Proceeding recursively over all periodic points for  $g$  we obtain a countable collection  $(\mathcal{A}_k)_k$  of open and dense sets in  $\mathcal{V}$  as above. In consequence, the subset

$$\mathcal{A}_\infty = \bigcap_{k \geq 0} \mathcal{A}_k \subset \mathcal{V}$$

is a residual subset of  $\mathcal{V}$  and for any  $\varphi \in \mathcal{A}_\infty$  it holds that:

- i. the periodic fibers are dense
- ii. for every  $k$  and  $0 \leq i < \pi(p_k)$  there exists a dense subset  $F_{i,k} \subset I_0$  such that all  $x \in F_{i,k}$  is in the basin of a periodic attractor for the multimodal map  $f_{i,k}$  and, consequently,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(p_k, x) \frac{\partial}{\partial x} \right\| < 0 \quad \text{for all } x \in F_{i,k} \subset I_0.$$

Since the set  $\cup \{ \{g^i(p_k)\} \times F_{i,k} : k \geq 1 \text{ and } 0 \leq i < \pi(p_k) \}$  is dense in  $\Lambda$  then properties (1), (2) and (3) in Theorem C are a consequence of the previous hyperbolicity reasoning above. Since  $\mathcal{V}$  is a set of generalized Viana maps then property (4) is a direct consequence of Theorem A. This finishes the proof of the theorem.  $\square$

## 7. SRB MEASURES FOR HYPERBOLIC GENERALIZED VIANA-MAPS

This section is devoted to prove Theorem D. Consider the skew-products

$$\begin{aligned} \varphi = \varphi_{a,\alpha} : \quad (0, 1] \times \mathbb{R} &\rightarrow (0, 1] \times \mathbb{R} \\ (\theta, x) &\mapsto (g(\theta), f_{a,\alpha}(\theta, x)) \end{aligned}$$

where  $g$  is a Markov expanding map on  $(0, 1]$  and  $f_{a,\alpha}(\theta, x) = a + \alpha \sin(2\pi\theta) - x^2$  for some parameters  $a \in (0, 2]$  and  $\alpha > 0$ . It follows from [GS97] that the set  $\mathcal{A} \subset (0, 2]$  of parameters  $a$  such that  $Q(x) = a - x^2$  is hyperbolic is open and dense in  $(0, 2]$ . Let  $a \in (0, 2) \cap \mathcal{A}$  be fixed.

It is not hard to check that if  $q$  is the fixed point of  $Q$  with largest absolute value then there exists a closed interval  $I_0 \subset [q, -q]$  for which we have the strict inclusion  $Q(I_0) \subset I_0$ . Moreover, by hyperbolicity  $Q$  has a periodic attracting point  $p$  of period  $\pi(p) \geq 1$  and  $I_0 = K \cup B(p)$ , where  $B(p)$  is the topological basin of attraction of  $p$  and  $K$  is an invariant Cantor set such that  $Q|_K$  is expanding. In particular,  $\varphi((0, 1] \times I_0) \subset (0, 1] \times I_0$  provided that  $\alpha > 0$  is small. Since  $\mathcal{A}$  is open and  $\alpha > 0$  is assumed small enough then it follows from the structural stability property that all quadratic maps  $f_\theta(x) = a + \alpha \sin(2\pi\theta) - x^2$  are hyperbolic and topologically conjugated to the quadratic map  $Q(x) = a - x^2$ : for every  $\theta \in (0, 1]$  there exists an homeomorphism  $h_\theta$  that is  $C^0$ -close to identity, varies continuously with  $\theta$  and such that  $f_\theta \circ h_\theta = h_\theta \circ Q$ . In consequence,  $p_\theta = h_\theta(p)$  is an attracting periodic point of period  $\pi(p)$  and  $K_\theta = h_\theta(K)$  is an invariant expanding Cantor set for the quadratic map  $f_\theta$ . We first prove that similar features to this one dimensional case also hold for the skew-product  $\varphi$ .

**Proposition 7.1.** *If  $\alpha > 0$  is small then there exists a  $\varphi$ -invariant set  $\mathcal{G} \subset (0, 1] \times I_0$  and there exists  $c > 0$  so that for all  $(\theta, x) \in \mathcal{G}$*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial x} \right\| \leq -c < 0$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial \theta} \right\| \geq c > 0.$$

*Proof.* Through the proof we assume without loss of generality that  $p$  is fixed for  $Q$  since otherwise we just consider the map  $Q^{\pi(p)}$ . By continuity, there exists  $\hat{\lambda} < \lambda < 1$  and a compact neighborhood  $\mathcal{U} \subset I_0$  of  $p$  such that  $|Q'(x)| \leq \hat{\lambda} < 1$  for all  $x \in \mathcal{U}$ . Hence, if  $\alpha > 0$  is small enough then  $p_\theta \in \mathcal{U}$ , we obtain the inclusion  $f_\theta(\mathcal{U}) \subsetneq \mathcal{U}$  and  $|f'_\theta(x)| \leq \lambda < 1$  for all  $\theta \in (0, 1]$ . By the chain rule

$|(f_{g^{n-1}(\theta)} \circ \cdots \circ f_{g(\theta)} \circ f_\theta)'(x)| \leq \lambda^n$  for all  $x \in \mathcal{U}$  and  $n \geq 1$ . Now we notice that  $\varphi((0, 1] \times \mathcal{U}) \subsetneq (0, 1] \times \mathcal{U}$  and consider the  $\varphi$ -invariant set

$$\mathcal{G} := \bigcap_{n \geq 0} \varphi^n((0, 1] \times \mathcal{U}).$$

Since  $(0, 1] \times \mathcal{U}$  is a compact set of  $(0, 1] \times I_0$  in the induced topology then  $\mathcal{G}$  is a compact set of  $(0, 1] \times I_0$  and has non-empty intersection with each vertical segment. Furthermore, the vertical diameter of each sequence of sets  $\varphi^n(\omega_n \times \mathcal{U})$ ,  $\omega_n \in \mathcal{P}^{(n)}$  is bounded from above by  $\lambda^n \text{diam}(\mathcal{U})$  converges to zero as  $n \rightarrow \infty$  tends to infinity. Finally, observe that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial \theta} \right\| \geq \log d > 0$$

and, by construction,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left\| D\varphi^n(\theta, x) \frac{\partial}{\partial x} \right\| &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log |(f_{g^{n-1}(\theta)} \circ \cdots \circ f_{g(\theta)} \circ f_\theta)'(x)| \\ &\leq \log \lambda < 0 \end{aligned}$$

for all  $(\theta, x) \in \mathcal{G}$ . This finishes the proof of the proposition.  $\square$

*Remark 7.2.* Let us mention that the set  $\bigcup \{(\theta, g_\theta(h^j(p))) : \theta \in (0, 1], 1 \leq j \leq \pi(p)\}$  of  $\pi(p)$  curves formed by the continuations of the periodic attractors  $p_\theta$  is not  $\varphi$ -invariant by the transversality condition of admissible curves. An interesting question is to determine whether  $\mathcal{G}$  can have nonempty interior.

Now we obtain the existence of a Cantor set of lines with expanding behavior in the complement of the topological basin of attraction of  $\mathcal{G}$ . More precisely, we prove the following:

**Proposition 7.3.** *If  $\alpha > 0$  is small there exists a  $\varphi$ -invariant set  $K \subset (0, 1] \times I_0$  so that the following properties hold:*

- (1)  $K_\theta = K \cap (\{\theta\} \times I_0)$  is a zero Lebesgue measure Cantor set for all  $\theta \in (0, 1]$ ;
- (2) The function  $(0, 1] \rightarrow \mathcal{P}(I_0)$  given by  $\theta \mapsto K_\theta$  is continuous, where  $\mathcal{P}(I_0)$  is endowed with the Hausdorff topology;
- (3) there exists  $c > 0$  such that for every  $(\theta, x) \in K$  and  $v \in \mathbb{R}^2$

$$\|D\varphi^n(\theta, x)v\| \geq e^{cn} \quad \text{for all } n \text{ large.}$$

*Proof.* Let  $\mathcal{U}$  be as in the proof of Proposition 7.1 and for every  $\theta \in (0, 1]$  set

$$K_\theta := \left\{ x \in \{\theta\} \times I_0 : f_\theta^n(x) \notin \text{int}(\mathcal{U}), \forall n \geq 0 \right\} = \bigcap_{n \geq 0} f_\theta^{-n}(I_0 \setminus \text{int}(\mathcal{U})),$$

where  $f_\theta^n = f_{g^{n-1}(\theta)} \circ \cdots \circ f_{g(\theta)} \circ f_\theta$ . Set also  $K = \{(\theta, x) \in (0, 1] \times I_0 : x \in K_\theta\}$ . It is clear from the construction that  $K$  is a compact  $\varphi$ -invariant set. Moreover, since it follows from Singer theorem that the critical point 0 is in the basin of the periodic attractor  $p$  for  $Q$  then there exists  $N \geq 1$  such that  $f_\theta^N(0) \in \mathcal{U}$  for all  $\theta \in (0, 1]$ , provided that  $\alpha > 0$  is small. Therefore, there exists an open neighborhood  $V$  of the critical region  $\{x = 0\}$  such that  $\varphi^N(V) \subset (0, 1] \times \mathcal{U}$ .

The later shows that all the iterates  $\{\varphi^n(\theta, x)\}_n$  of points  $(\theta, x) \in K$  avoid the neighborhood  $V$  of the critical region. Thus, Proposition 3.6(5) implies that the restriction  $f|_K$  is expanding and that  $K_\theta$  cannot have positive Lebesgue measure.

We proceed to prove now that  $K_\theta$  is a Cantor set. Given any  $\theta \in (0, 1)$  we have  $K_\theta = \cap_{n \geq 0} K_{\theta, n}$  where each  $K_{\theta, n} = \cap_{j=0}^n f_\theta^{-j}(I_0 \setminus \text{int}(\mathcal{U}))$  is the union of intervals. In particular  $K_\theta$  is a perfect set. Assuming, by contradiction, that there exists an interval  $J \subset K_\theta$ . Then Proposition 3.6(5) together with the Mean Value Theorem is enough to prove that there exists  $m \geq 1$  such that  $f_\theta^m(J) = I_0$  and consequently  $K \cap \mathcal{U} \supset f_\theta^m(J) \cap \mathcal{U} \neq \emptyset$ . The contradiction proves that  $K_\theta$  is totally disconnected. In consequence, each  $K_\theta$  is a Cantor set of zero Lebesgue measure.

Finally, notice that for all  $n \in \mathbb{N}$  the map  $\theta \mapsto K_{\theta, n}$  is continuous in the Hausdorff topology in view of the continuity of the map  $\theta \mapsto f_\theta^n$ . Since  $K_\theta = \cap_{n \geq 0} K_{\theta, n}$  and This finishes the proof of the second assertion and the proposition.  $\square$

It follows from the two last propositions that Lebesgue almost every point in  $(0, 1] \times I_0$  is in the topological basin of attraction of the set  $\mathcal{G}$ . So, we consider the restriction  $\varphi|_{\mathcal{G}}$  and proceed to prove that it admits a unique SRB measure. Let  $\mu$  be the unique absolutely continuous invariant measure for  $g$  and consider the sequence of probability measures given by

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} \varphi_*^j(\text{Leb}|_{(0,1] \times \mathcal{U}}), \quad n \geq 1.$$

We claim that  $(\nu_n)_n$  is convergent to an  $\varphi$ -invariant, ergodic probability measure  $\nu$  whose basin of attraction covers Lebesgue almost every point in the attractor. This will prove that  $\varphi$  has a unique SRB measure. On the one hand, if one considers the projection  $\pi_1 : (0, 1] \times I_0 \rightarrow (0, 1]$  the sequence  $(\pi_1)_* \nu_n = \frac{1}{n} \sum_{j=0}^{n-1} g_*^j(\text{Leb}|_{(0,1]})$  is convergent to the unique  $g$ -invariant and ergodic absolutely continuous probability measure  $\mu$ .

On the other hand, the uniform contraction on  $\mathcal{U}$  under iteration by  $f_\theta^n$  implies that for any continuous observable  $G$ ,  $\varepsilon > 0$  and points  $x, y \in \{\theta\} \times \mathcal{U}$  the Birkhoff averages satisfy  $|\frac{1}{n} \sum_{j=0}^{n-1} G \circ \varphi^j(\theta, x) - \frac{1}{n} \sum_{j=0}^{n-1} G \circ \varphi^j(\theta, y)| < \varepsilon$  provided that  $n$  is large (depending only on  $g$  and  $\lambda$ ). Therefore, the functional  $\Psi : C((0, 1] \times \mathcal{U}) \rightarrow \mathbb{R}$  given by

$$\Psi(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \max_{x \in \mathcal{U}} [G \circ \varphi^j(\theta, x)] d\mu(\theta) \quad (7.1)$$

is well defined and clearly continuous. By Riesz representation theorem there exists a probability measure  $\nu$  with  $\text{supp}(\nu) \subset (0, 1] \times \mathcal{U}$  and such that  $\Psi(G) = \int G d\nu$  for all continuous function  $G$ . Furthermore, since  $\mu$  is  $g$ -invariant and ergodic the limit (7.1) is almost everywhere constant in  $\theta$  and consequently

$$\int G d\nu = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int G \circ \varphi^j(\theta, x) d\mu(\theta)$$

for  $\mu \times \text{Leb}$ -almost every  $(\theta, x) \in (0, 1] \times \mathcal{U}$ . In particular, this proves that  $\nu$  is  $\varphi$ -invariant and ergodic. Using that Lebesgue almost every  $(\theta, x)$  is eventually mapped in the set  $(0, 1] \times \mathcal{U}$  it follows that the basin  $B(\nu)$  is a full Lebesgue measure set on the attractor. This proves our claim. Finally just observe by construction that  $\nu$  has a negative Lyapunov exponent in the direction  $\partial/\partial x$  and a positive Lyapunov exponent in direction  $\partial/\partial \theta$ , which completes the proof of Theorem D.

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